11862

ASS NOTE 17
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APPROXIMATIONS IN THE INFINITE POPULATION MODEL OF THE ARPANET SATELLITE SYSTEM

1. Introduction

In ASS Note 12, a "Markovian" model for the satellite channel with an infinite user population* was studied in detail. The relevant questions regarding channel throughput rate and packet delays in relation to traffic levels and random retransmission delays** were studied. However, such results were based upon a very strong assumption (A.4 in ASS Note 12): The traffic arriving to the satellite channel is assumed to be a Poisson process. This assumption implies two things: (A.4.1) the number of arrivals to a time slot is Poisson distributed and (A.4.2) the arrivals to all time slots are independent. Such a strong assumption has invited some critical comments. In this note, for a large k we relax assumption (A.4) to

(A.4') arrivals to different time slots which lie within k slots of each other are independent.

Using (A.4'), we prove that the traffic arriving to a time slot indeed has a Poisson distribution as $k \to \infty$. We also elaborate upon the approximation involved in this mathematical model. Lastly, an improvement is made to the results of ASS Note 12: $q \triangleq \text{Prob}[\text{successful transmission} \mid \text{new packet}]$ will be calculated in a fashion which is more consistent with the "Markovian" assumption of this model. As a result, we obtain slightly more optimistic results. The overall behavior of the model is unchanged. Also, all the equations given in ASS Note 12 remain valid. Proofs for propositions 1 and 2 in this note are

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*This assumption gives pessimistic results since in this case, every packet will interfere with every other packet.

**We again assume that the random retransmission delay is uniformly distributed over k slots.

given in Appendix I. In Appendix II, we give some comments on Bob Metcalf's ASS Note 16 and suggest an alternate solution.

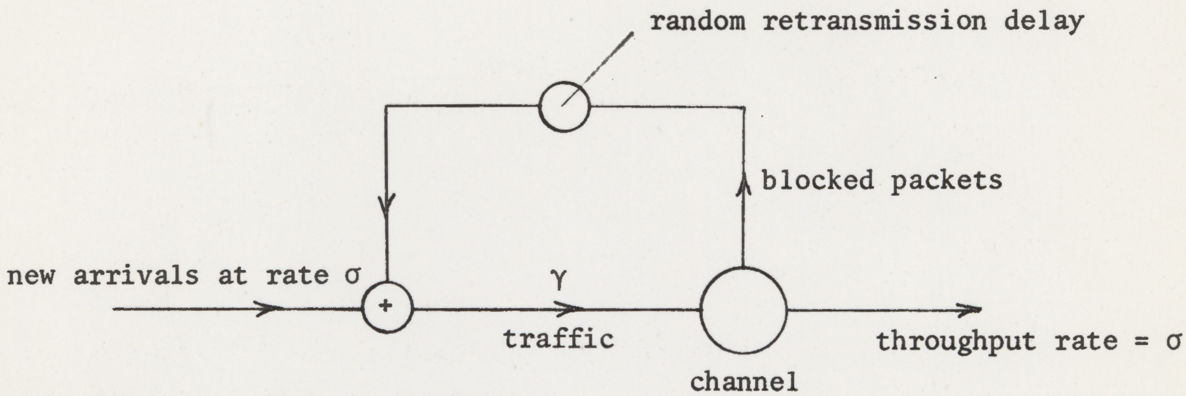

Fig. 1. The Model

2. Distribution of Traffic Arrivals to a Time Slot

Any dependence among arrivals to different time slots is caused by previous collisions at the satellite channel. Given two time slots within k slots of each other, if $kT < d$,* there will be no direct retransmitted packets from one slot to the other. Hence, any dependence arises when collided packets from the past are retransmitted to the given slots. Such dependence is believed to be small and decreases as k increases.

Define

$C_n \triangleq$ number of traffic arrivals to the n^{th} time slot

$$p_i^{(n)} \triangleq \mathrm{Prob}[C_n = i] \qquad i = 0,1,2, \ldots$$

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*T is the slot length and d is the round trip propagation delay. For a satellite channel d is large and $kT < d$ is true in almost all practical situations. However, in the theoretical limit as $k \uparrow \infty$ we must assume that $T \downarrow 0$ if we are to have $kT < d$.

$$Q_n(z) \triangleq \sum_{i=0}^{\infty} z^i P_i^{(n)}$$

$$P_i \triangleq \lim_{n\to\infty} P_i^{(n)}$$

$$Q(z) \triangleq \lim_{n\to\infty} Q_n(z)$$

$$\gamma \triangleq \sum_{i=1}^{\infty} iP_i = \text{average channel traffic/slot}$$

$$v_i \triangleq \text{Prob[number of external new arrivals to a time slot} = i]$$

$$V(z) \triangleq \sum_{i=0}^{\infty} z^i v_i$$

$$\sigma \triangleq \sum_{i=1}^{\infty} iv_i = \text{average throughput/slot}$$

Without loss of generality, let $T = 1$

Proposition 1

Assuming (A.4') and under stationary conditions, $Q(z)$ is given by

$$Q(z) = \left[\frac{P_1}{k}(1 - z) + Q(1 - \frac{1}{k} + \frac{z}{k})\right]^k V(z) \tag{1}$$

If the external new arrivals are Poisson distributed, the traffic arrivals are Poisson distributed in the limit as $k \to \infty$, and

$$\lim_{k\to\infty} Q(z) = e^{-\gamma(1-z)} \tag{2}$$

$$\lim_{k\to\infty} \sigma = \gamma e^{-\gamma} \tag{3}$$

Eq. (1) is quite general requiring only assumption (A.4'). The external new arrivals can have any distribution. An explicit solution for $Q(z)$ cannot be obtained except when $k \to \infty$ or in some uninteresting cases, one of which we give here as an example.

Example $k = 1$

We have from Eq. (1)

$$P_0 + P_1 z + P_2 z^2 + \ldots = (P_0 + P_1 + P_2 z^2 + \ldots)(v_0 + v_1 z + v_2 z^2 + \ldots) \quad (4)$$

Equating coefficients for the first two terms, we obtain

$$P_1 = \frac{1 - v_0}{v_0} P_0$$

$$P_1 = \frac{v_1}{1 - v_1} P_0$$

which imply that Eq. (4) is inconsistent unless $v_1 = 1 - v_0$. Hence, as $n \to \infty$ the stationary limit $Q(z)$ does not exist unless Prob[number of new arrivals to a slot ≥ 2] = 0. In that case,

$$P_0 = v_0 \qquad P_1 = v_1$$

and from Eq. (4) $Q(z) = (P_0 + P_1)(v_0 + v_1 z) = v_0 + v_1 z = V(z)$ which is trivially true.

3. Approximations in the Model

Since a mathematical model is only as good as its assumptions, in this section we shall briefly go over the model approximations. Consider the arrival of all packets (new or retransmitted) to the satellite channel as a discrete time stochastic process. To give an exact characterization of the process, we need the joint probabilities $P(C_{n_1} = i_1, C_{n_2} = i_2, \ldots, C_{n_\ell} = i_\ell)$ for all ℓ, $n_1, \ldots, n_\ell, i_1, \ldots, i_\ell \geq 0$. From these we get both the distri-

bution of the number of arrivals to any time slot as well as the time correlation among arrivals to all time slots. A first-order approximation (as done in the early ASS Notes) is to assume Poisson distributed arrivals to each time slot and independence among arrivals to all time slots. The Poisson assumption has been shown by the previous proposition to be good when k is large. The independence assumption, however, has been weakened to that of "local independence" among arrivals to k adjacent slots. Time correlation is assumed to be "Markovian," i.e., the process has a finite memory about its past. Such a model is believed to portray fairly accurately the behavior of the real system without getting too involved in mathematical analysis.

4. An Improvement to Results in ASS Note 12

The "Markovian" time correlation assumption has the added advantage of bringing out the effects of the random retransmission delay (assumed to be uniformly distributed over k slots). Recall from ASS Note 12,

$$q = \text{Prob}[\text{successful transmission} \mid \text{new packet}]$$

$$q_t = \text{Prob}[\text{successful transmission} \mid \text{previously blocked}]$$

The effect of k was built into q_t, whereas q was taken to be $e^{-\gamma T}$. Since the event [new packet] gives us added information, the effect of k should also be built into q to be consistent. In a similar manner to the derivation of q_t, we now see that

$$q = q_i^k \, e^{-\sigma} \tag{5}$$

where, referring to Fig. 4 in Appendix I and for $T = 1$

$$q_i = \text{Prob}[\text{no retransmissions from the } (n\text{-}d\text{-}i)^{\text{th}} \text{ slot to slot } n]$$

$$= e^{-\gamma/k} + \frac{\gamma}{k} e^{-\gamma}$$

and

$$\frac{\sigma}{\gamma} = \frac{q_t}{q_t + 1 - q}$$

Proposition 2

In the limit as $k \to \infty$, q converges to $e^{-\gamma T}$.

5. Final Remarks

In this note we have examined the approximations involved in our mathematical model for the infinite population satellite system. We came to the conclusion that the Poisson and "Markovian" assumptions will serve our purpose in approximating the behavior of the traffic arrival process to the satellite channel.

An improvement was made to give q a new value to reflect the "Markovian" property of the model. In Figs. 3 and 4 we plot q and q_t and the channel throughput rate σ as functions of k and γ. Comparing these with the corresponding plots in ASS Note 12, it can be seen that we now have more optimistic results for small values of k and γ. Also, all the equations in ASS Note 12 remain valid with the new value for q.

FIG. 2 Prob [Success/ blocked] & Prob [Success/new] vs. k

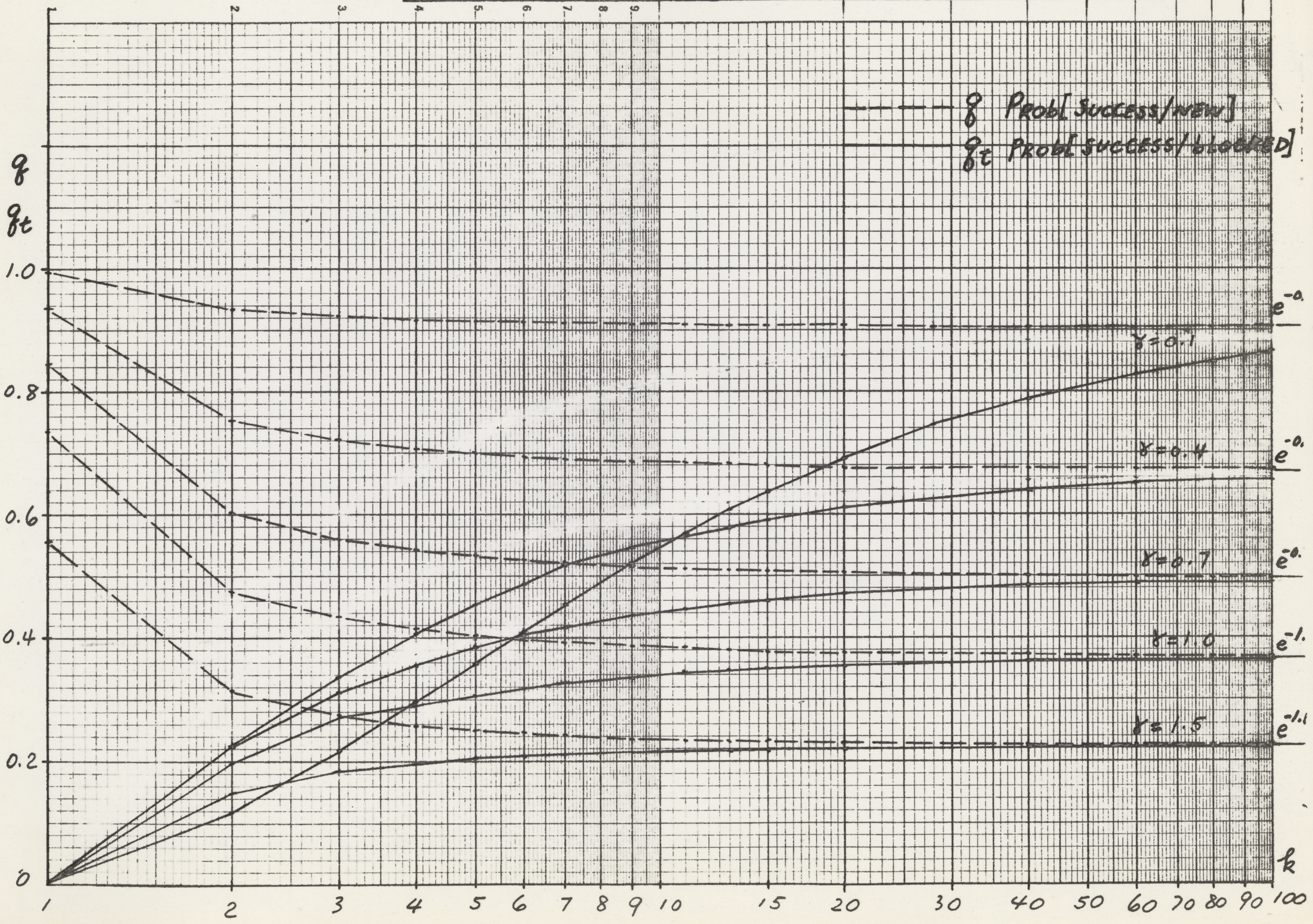
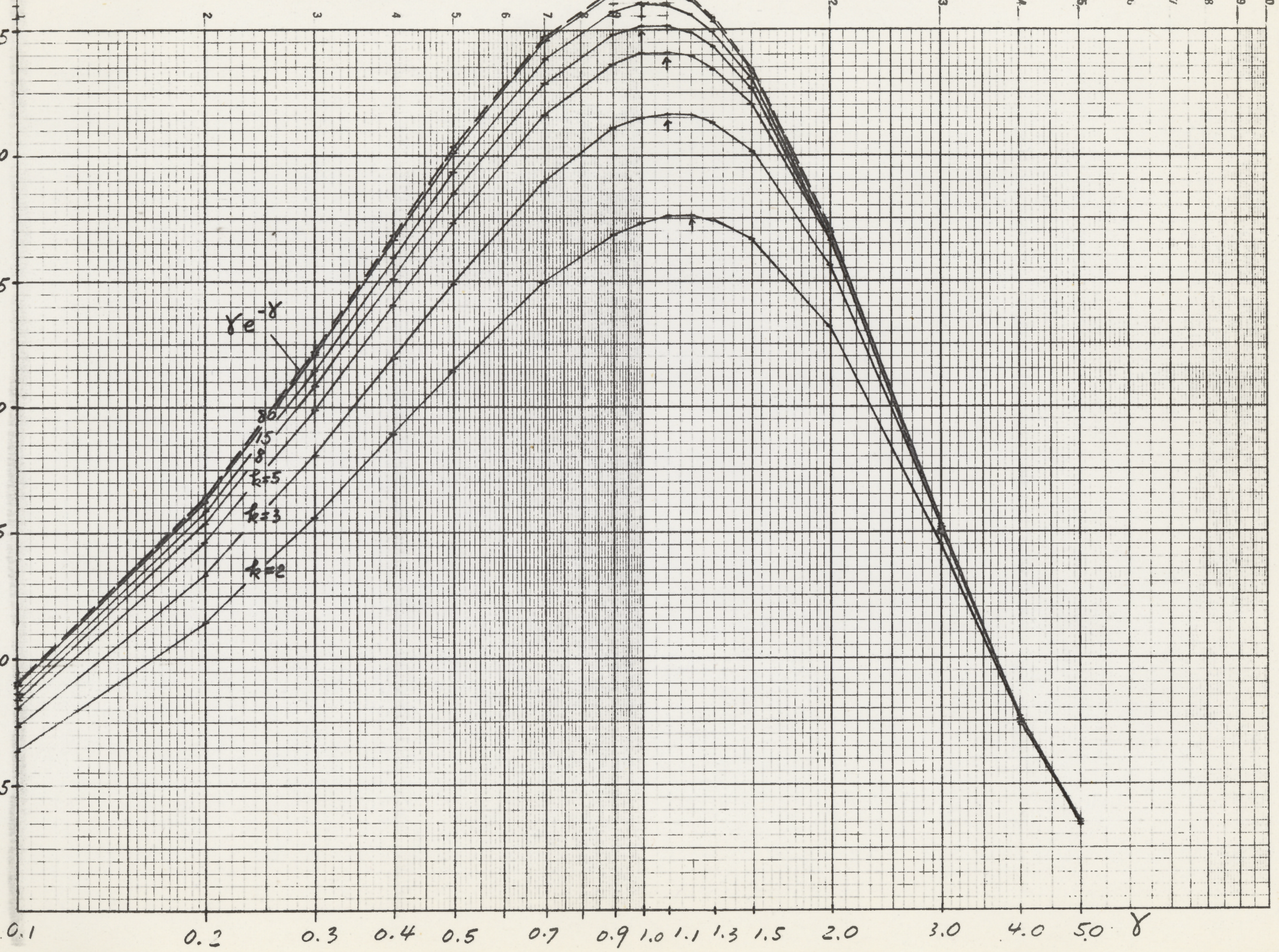


FIG. 3

Throughput RATE σ vs. γ



APPENDIX I

Proof for Proposition 1

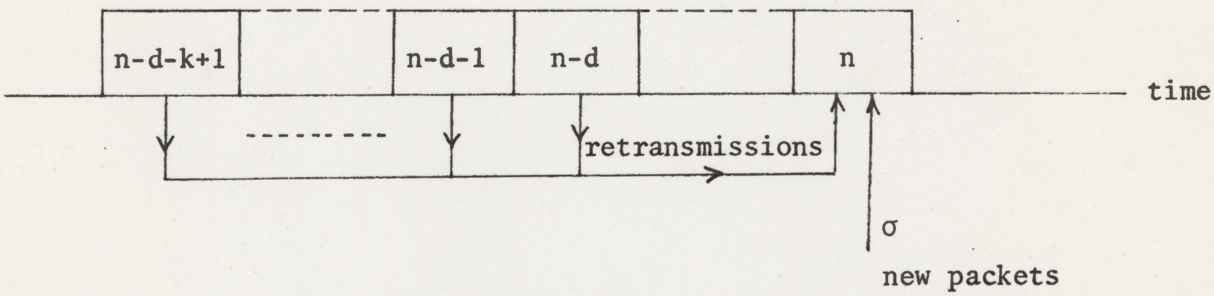


Fig. 4. Traffic Arrivals to a Time Slot

Let $h_i^{(n-d-j)}$ = Prob[i retransmissions from slot (n-d-j) to slot n]

$j = 0,1, \ldots, k-1$

$$\hat{Q}_{n-d-j}(z) \triangleq \sum_{i=0}^{\infty} z^i \, h_i^{(n-d-j)}$$

$$= p_0^{(n-d-j)} + p_1^{(n-d-j)} + \sum_{m=2}^{\infty} \left(1 - \frac{1}{k}\right)^m p_m^{(n-d-j)}$$

$$+ z \left[\sum_{m=2}^{\infty} \binom{m}{1} \left(\frac{1}{k}\right) \left(1 - \frac{1}{k}\right)^{m-1} p_m^{(n-d-j)} \right]$$

$$+ \sum_{i=2}^{\infty} z^i \left[\sum_{m=i}^{\infty} \binom{m}{i} \left(\frac{1}{k}\right)^i \left(1 - \frac{1}{k}\right)^{m-i} p_m^{(n-d-j)} \right]$$

$$= \sum_{i=0}^{\infty} z^i \left[\sum_{m=i}^{\infty} \binom{m}{i} \left(\frac{1}{k}\right)^i \left(1 - \frac{1}{k}\right)^{m-i} p_m^{(n-d-j)} \right]$$

$$+ p_1^{(n-d-j)} - \frac{p_1^{(n-d-j)}}{k} z - \left(1 - \frac{1}{k}\right) p_1^{(n-d-j)}$$

$$= \frac{P_1^{(n-d-j)}}{k}(1 - z) + \sum_{m=0}^{\infty} \sum_{i=0}^{m} z^i \binom{m}{i} \left(\frac{1}{k}\right)^i \left(1 - \frac{1}{k}\right)^{m-i} P_m^{(n-d-j)}$$

$$= \frac{P_1^{(n-d-j)}}{k}(1 - z) + \sum_{m=0}^{\infty} \left(1 - \frac{1}{k} + \frac{z}{k}\right)^m P_m^{(n-d-j)}$$

$$= \frac{P_1^{(n-d-j)}}{k}(1 - z) + Q_{n-d-j}\left(1 - \frac{1}{k} + \frac{z}{k}\right)$$

Assuming (A.4') and since the external new arrivals are assumed to be independent of the state of the system,

$$Q_n(z) = \left(\prod_{j=0}^{k-1} \hat{Q}_{n-d-j}(z)\right) V(z) \tag{I.1}$$

Under stationary conditions as $n \to \infty$,

$$Q(z) = [\hat{Q}(z)]^k \, V(z)$$

$$= \left[\frac{P_1}{k}(1 - z) + Q\left(1 - \frac{1}{k} + \frac{z}{k}\right)\right]^k V(z)$$

where $\hat{Q}(z) \triangleq \lim_{n\to\infty} \hat{Q}_{n-d-j}(z)$

Consider,

$$Q\left(1 - \frac{1}{k} + \frac{z}{k}\right) = \sum_{i=0}^{\infty} P_i \left(1 - \frac{1}{k} + \frac{z}{k}\right)^i$$

$$= P_0 + \sum_{i=1}^{\infty} P_i \left[1 - \frac{i}{k}(1 - z)\right] + o\left(\frac{1}{k}\right) \qquad \text{where } \lim_{x\to 0} \frac{o(x)}{x} \to 0$$

$$= \sum_{i=0}^{\infty} P_i - \frac{(1-z)}{k} \sum_{i=1}^{\infty} i\, P_i + o\left(\frac{1}{k}\right)$$

$$= 1 - \frac{\gamma(1-z)}{k} + o\left(\frac{1}{k}\right)$$

$$Q(z) = \left[1 - \frac{\gamma(1-z)}{k} + \frac{P_1(1-z)}{k} + o\left(\frac{1}{k}\right)\right]^k V(z) \qquad \text{(I.2)}$$

$$\lim_{k\to\infty} Q(z) = e^{-\gamma(1-z)+P_1(1-z)} V(z)$$

Now assuming the external new arrivals to be Poisson distributed,

$$V(z) = e^{-\sigma(1-z)}$$

$$\lim_{k\to\infty} Q(z) = e^{-\gamma(1-z)+P_1(1-z)} e^{-\sigma(1-z)}$$

where P_1 = Prob[exactly 1 arrival to the satellite channel in a time slot]

= Prob[a packet obtains successful transmission]

= throughput rate of the channel.

Under stationary conditions, the channel throughput rate is equal to the channel input rate,

$$\therefore, \quad P_1 = \sigma$$

Hence,

$\lim_{k\to\infty} Q(z) = e^{-\gamma(1-z)}$ which is the z-transform of a Poisson distribution with arrival rate = γ per slot. An immediate consequence is,

$$\lim_{k\to\infty} \sigma = P_1 = \gamma e^{-\gamma}$$

Q.E.D.

Proof for Proposition 2

Assume $q = e^{-\gamma T}$, we want to show that Eq. (5) holds in the limit as $k \to \infty$

Let $T = 1$

$$\lim_{k\to\infty} \text{RHS} = \lim_{k\to\infty} \left[1 + \frac{\gamma}{k} e^{-\gamma+\gamma/k}\right]^k \left[e^{-\gamma/k}\right]^k e^{-[q_t/(q_t+1-q)]\gamma}$$

From Proposition 1 in ASS Note 12, we have

$$\lim_{k\to\infty} q_t = e^{-\gamma} \quad \text{given that} \quad \lim_{k\to\infty} q = e^{-\gamma}$$

Hence,

$$\lim_{k\to\infty} \text{RHS} = e^{-\gamma(1-e^{-\gamma})} e^{-[e^{-\gamma}/(e^{-\gamma}+1-e^{-\gamma})]\gamma}$$

$$= e^{-\gamma}$$

$$= \lim_{k\to\infty} \text{LHS}$$

Q.E.D.

APPENDIX II

Comments on ASS Note 16 by Bob Metcalfe

In ASS Note 16, with Q defined to be the steady-state time-average of the number of terminals with packets ready and X = probability of transmission at a terminal with a ready packet,* the following equations are given (in Fortran notation):

W = probability of throughput in a slot
= Q * X * (1 - X) ** (Q - 1) (II.1)

U = probability of traffic in a slot
= 1 - (1 - X) ** Q (II.2)

X_{opt} = value of X which optimizes throughput rate for a given Q
= 1/Q (II.3)

We discussed these equations with Metcalfe and pointed out that his use of the average value Q, was improper since he was dealing with non-linear functions of Q. We compared his results with ours (given below) and agreed that as $X \to 0$, both his and ours yield the same results for Eqs. (II.1) and (II.2); however, Eq. (II.3) must be modified as we show below.

If we define the random variables

q = instantaneous number of terminals with packets ready

w = instantaneous probability of throughput in a slot

u = instantaneous probability of traffic in a slot

and observe that if Q, W and U are the instantaneous conditional values of the random variables q, w and u, then the above equations are indeed correct.

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*In this case the packet retransmission delay is geometric rather than uniformly distributed.

However, in this case only the instantaneous value of the throughput rate in the next slot is maximized with $X_{opt} = 1/q$. It is not clear if the steady-state throughput rate is also maximized. Also, system overhead will be significant since q has to be measured and the value X_{opt} adjusted at each slot.

It is assumed in ASS Note 16 that the slot duration D is long enough so that a sender will know the success of a transmission before the start of the next slot. This assumption may be valid for a radio channel, but is unrealistic for a satellite channel due to the large round trip propagation delay. In the latter case the set of terminals with packets ready may be very different in 2 adjacent slots and it is clear that we need more than just the first moment of q to obtain the optimal value of X.

We propose the following alternate solution. The value of X is adjusted periodically. Assuming that the probability distribution (actually we need only the first 2 moments) of q in the current period can be measured and it will approximate the probability distribution of q in the next period (this assumes the length of a period to be short), we rewrite Eqs. (II.1) and (II.2) as,

$$E[w] = \sum_{i=0}^{N} i\, X(1 - X)^{i-1} P_i \tag{II.4}$$

$$E[u] = 1 - \sum_{i=0}^{N} (1 - X)^{i} P_i \tag{II.5}$$

where N = total number of terminals

$P_i = \text{Prob}[q = i]$

Putting $\frac{\partial E[w]}{\partial X} = 0$, we have

$$\sum_{i=0}^{N} i(1 - iX)(1 - X)^{i-2} P_i = 0$$

Assuming that X is so small that we may neglect higher order terms in the expansion of $(1 - iX)(1 - X)^{i-2}$, we have

$$\sum_{i=0}^{N} i\Big(1 - 2(i - 1)X\Big) P_i = 0$$

$$\overline{q} - 2X(\overline{q^2} - \overline{q}) = 0 \qquad \text{where} \quad \overline{q^n} = E[q^n]$$

$$X_{opt} = \frac{\overline{q}}{2(\overline{q^2} - \overline{q})} = \frac{1}{2(\overline{q^2}/\overline{q} - 1)} \tag{II.6}$$

The above is the optimal value of X for the static optimization of $E[w]$ given that we know the first two moments of q.

From the above discussion, we see that if we know the exact value of q (the number of terminals with ready packets), then $X = 1/q$ is optimal. However, if we do not know the exact value of q, then the use of $\overline{q}$., viz., $X = 1/\overline{q}$ is not optimal. Instead

$$X_{opt} = \frac{1}{2(\overline{q^2}/\overline{q} - 1)}$$

under the approximations of our expansion.